Math 4200
Wednesday September 30

2.3 Section 2.2 results reimagined more rigorously and more generally.

Announcements: Quiz today.
may or may not do the "connected" = "path connected" discussion today.

~~Warm-up exercise:~~ graded hw3
~ solutions are on CANVAS.

In section 2.2 we have carefully shown:

Theorem If A is a connected open subset of $\mathbb{C}$, $f : A \to \mathbb{C}$ continuous, then contour integrals $\int_\gamma f(z)\, dz$ are path independent if and only if $\exists$ an antiderivative $F(z)$ to $f(z)$ (and F can be defined with contour integrals).

rigorous !!

We have less carefully proven that

Theorem If A is open and *simply connected*, $f : A \to \mathbb{C}$ analytic and C^1, then $\int_\gamma f(z)\, dz$ are path-independent, so there exist antiderivatives $F(z)$.

issues:

(i) We defined a domain to be *simply connected* to mean that it has no holes. This is visually appealing but imprecise and hard to describe analytically.

(ii) We did not show path independence for all piecewise C^1 paths - we just drew a few pictures which are not representative of all possible configurations: we assumed they crossed in such a way as to create a finite number of subdomains on which to apply Green's Theorem.

(iii) We had to assume $f \in C^1$ for Green's Theorem, whereas it turns out we will only need that f is pointwise analytic on A for Cauchy's Theorem and antidifferentiation theorems (and this is useful to know).

The goal of section 2.3 is to deal with these issues, and also to prove precise and stronger versions of the Deformation Theorem, about when you can change the contour curves for $\int_\gamma f(z)\, dz$ without changing the actual value of the contour integrals. Along the way we introduce the notion of *homotopy,* key to many areas of mathematics, especially ones that use *algebraic topology*. (See Wikipedia.)

2.3 first step: improved (but only local) antidifferentiation theorem:

Theorem Let $f : D(z_0; r) \to \mathbb{C}$ be analytic. Then $\exists\, F : D(z_0; r) \to \mathbb{C}$ such that $F' = f$ in $D(z_0; r)$.

Rectangle Lemma Let f, $D(z_0; r) = D$ be as above. Let $R = [a, b] \times [c, d] \subseteq D$ be a coordinate rectangle inside the disk. (i.e. $R = \{x + i\,y \mid a \le x \le b,\ c \le y \le d\} \subseteq D$.) Let $\gamma = \delta R$, oriented counterclockwise. Then

$$\int_\gamma f(z)\, dz = 0.$$

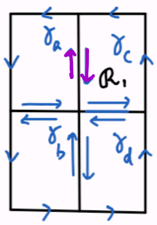
• (If f was C^1 we'd already know this result via Green's Theorem.)

proof: (Goursat):

to show $\int_\gamma f(z)\,dz = 0.$

p = perim of R
d = diagonal

Subdivide:

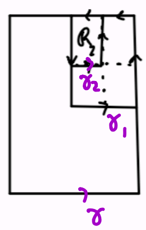


- $\int_\gamma f(z) = \int_{\gamma_a} f + \int_{\gamma_b} f + \int_{\gamma_c} + \int_{\gamma_d}$ (interior cancellation)

$\Rightarrow \left|\int_\gamma f(z)dz\right| \le |\int| + |\int| + |\int| + |\int|$ •

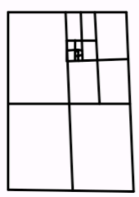
$\le 4 \left|\int_{\gamma_1} f(z)dz\right|$ where $\gamma_1 = \partial R_1$ has the contour integral with largest modulus •

R_1 perimeter $= p/2$ •
diag $= d/2$



- pick $R_2 \subset R_1$
  $\partial R_2 = \gamma_2$ s.t.
  - $\left|\int_{\gamma_1} f(z)dz\right| \le 4\left|\int_{\gamma_2} f(z)dz\right|$

$\left|\int_\gamma f(z)dz\right| \le 4\left|\int_{\gamma_1} f(z)dz\right| \le 4^2\left|\int_{\gamma_2} f(z)dz\right|$



- induct $R \supset R_1 \supset R_2 \supset \dots \supset R_k$
- $\left|\int_\gamma f(z)dz\right| \le 4^k \left|\int_{\gamma_k} f(z)dz\right|$
  - p_k = perimeter of $R_k = p\,2^{-k}$
  - d_k = diag length " $= d\,2^{-k}$
- $\{R_k\}$ nested, diameters $d_k \to 0$

$\Rightarrow \bigcap_k R_k = z_0 \in R.$ (all our rectangles are closed)

↑ not the center of the disk we started with (oops!)

pick $z_k \in R_k$

by construction $\{z_k\}$ is Cauchy

$\Rightarrow \{z_k\} \to L \in \mathbb{C}$, ind. of seq. $\{z_k\}$.

R_k closed $\Rightarrow L \in R_k\ \forall k.$
$\quad\quad\quad\quad\quad\quad\quad\ \ \|$
$\quad\quad\quad\quad\quad\quad\quad\ z_0.$

punchline: f is analytic at z_0. Thus for z near z_0:

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + (z - z_0)\varepsilon(z - z_0)$$

where the error function

$$\varepsilon(z - z_0) \to 0 \quad \text{as } z \to z_0.$$

Let $\epsilon > 0$. Pick k such that the error satisfies

$$|\varepsilon(z - z_0)| \le \epsilon, \ \forall\ z \in R_k.$$

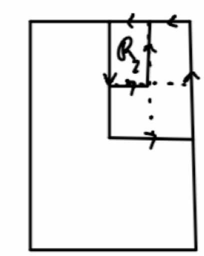
Now estimate

$$\int_{\gamma_k} f(z)\, dz = \int_{\gamma_k} f(z_0) + f'(z_0)(z - z_0)\, dz + \int_{\gamma_k} (z - z_0)\varepsilon(z - z_0)\, dz.$$

By the FTC, and if γ_k starts and ends at a point Q,

$$\int_{\gamma_k} f(z_0) + f'(z_0)(z - z_0)\, dz = f(z_0)z + f'(z_0)\frac{(z - z_0)^2}{2}\Big]_Q^Q = 0.$$

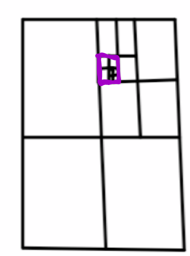
So



$$\int_{\gamma_k} f(z)\, dz = \int_{\gamma_k} (z - z_0)\varepsilon(z - z_0)\, dz$$

$$\left|\int_{\gamma_k} f(z)\, dz\right| \le \int_{\gamma_k} |(z - z_0)| |\varepsilon(z - z_0)|\, |dz|$$

$$\le d_k \epsilon p_k \le \epsilon\, 2^{-k} d\, 2^{-k} p = \epsilon\, 4^{-k} p\, d.$$



And we estimate the original contour integral,

$$\left|\int_{\gamma} f(z)\, dz\right| \le 4^k \left|\int_{\gamma_k} f(z)\, dz\right| \le 4^k \epsilon\, 4^{-k} p\, d = \epsilon\, p\, d.$$

Since this estimate is true for all ϵ,

$$\left|\int_{\gamma} f(z)\, dz\right| = 0$$

which proves the rectangle lemma.

Q.E.D.

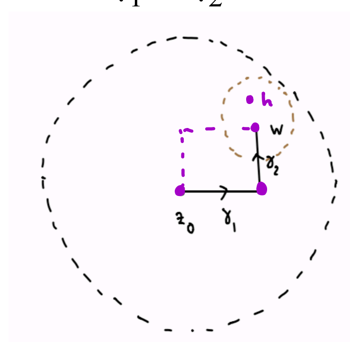
Now complete the proof of the local antidifferentiation theorem:

Theorem Let $f : D(z_0; r) \to \mathbb{C}$ be analytic. Then $\exists\, F : D(z_0; r) \to \mathbb{C}$ such that $F' = f$ in $D(z_0; r)$.

proof: Let $w \in D(z_0; r)$. Consider the closed rectangle $R(w)$ which has z_0 and w as opposite corners. (This rectangle will collapse into a line segment if $w - z_0$ is purely real or imaginary.) Let γ_1 be the real-direction curve from z_0 to $z_0 + \mathrm{Re}(w - z_0)$; let γ_2 be the imaginary direction path from $z_0 + \mathrm{Re}(w - z_0)$ to w, as indicated below. Note, depending on the relative location of z_0 and w, γ_1 may move in either the positive or negative real direction; γ_2 may move in either the positive or negative imaginary direction. Define

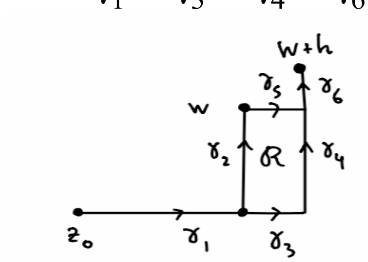
unlike in 2.2, we're using specific contours for our antiderivative.

- $$F(w) = \int_{\gamma_1 + \gamma_2} f(z)\, dz.$$

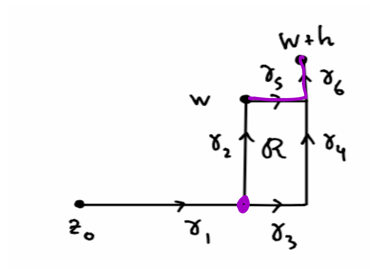


To show that $F'(w) = f(w)$ we will verify the affine approximation formula with error. Let $h \in D(w; r - |w - z_0|) \subseteq D(z_0; r)$. Then, for the contours indicated below, we see that

$$F(w + h) = \int_{\gamma_1 + \gamma_3 + \gamma_4 + \gamma_6} f(z)\, dz.$$



- $$F(w + h) - F(w) = \int_{\gamma_1 + \gamma_3 + \gamma_4 + \gamma_6} f(z)\, dz - \int_{\gamma_1 + \gamma_2} f(z)\, dz \quad .$$



So

$$F(w+h) - F(w) = \int_{\gamma_1+\gamma_3+\gamma_4+\gamma_6} f(z)\,dz - \int_{\gamma_1+\gamma_2} f(z)\,dz \quad ,$$

$$F(w+h) - F(w) = \int_{\gamma_3+\gamma_4+\gamma_6} f(z)\,dz - \int_{\gamma_2} f(z)\,dz$$

As the diagram indicates, the parallel curves γ_2, γ_4 and the parallel curves γ_3, γ_5 bound a rectangle (or line segment). And regardless of how which quadrant h is in, the curves $\gamma_3+\gamma_4$ and $\gamma_2+\gamma_5$ have the same initial and terminal points. So by the rectangle lemma,

$$\int_{\gamma_3+\gamma_4} f(z)\,dz = \int_{\gamma_2+\gamma_5} f(z)\,dz\,.$$

So

$$F(w+h) - F(w) = \int_{\gamma_2+\gamma_5+\gamma_6} f(z)\,dz - \int_{\gamma_2} f(z)\,dz = \int_{\gamma_5+\gamma_6} f(z)\,dz\,.$$

Proceed using the same strategy as we used in section 2.2, but with slightly different countours this time: We use FTC for the first term and estimate the second, error, term.

$$\int_{\gamma_5+\gamma_6} f(z)\,dz = \int_{\gamma_5+\gamma_6} f(w)\,dz + \int_{\gamma_5+\gamma_6} f(z)-f(w)\,dz$$

// FTC

$$= f(w)\,h + h\,\varepsilon(h)$$

where

$$\varepsilon(h) = \frac{1}{h}\int_{\gamma_5+\gamma_6} f(z)-f(w)\,dz$$

$|\varepsilon(h)| \le \frac{1}{|h|} \max_{\gamma_5+\gamma_6} |f(z)-f(w)|\; 2|h|$

and $\varepsilon(h)\to 0$ as $h\to 0$. In other words,

$$F(w+h) = F(w) + f(w)\,h + h\,\varepsilon(h)\,.$$

$$F'(w) = f(w)\,.$$

Q.E.D.

On Friday, we'll talk about homotopies, simply connected analysis's def + more.